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Blow-up of Solutions for Semilinear Parabolic Equations

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Section 1. Main results.

In this paper we consider the following initial boundary value problems

(IBVP1)
$$\begin{cases} u_t(x,t) = \Delta u + \lambda e^{u(x,t)} & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = \varphi(x) & \text{in } \Omega, \end{cases}$$

(IBVP2)
$$\begin{cases} u_t(x,t) = \Delta u + u^p(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = \varphi(x) & \text{in } \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ and $1 < p < (n+2)/(n-2)$ $(n \geq 3)$, $p > 1$ $(n=2)$.

Throughout the present paper we assume that an initial value φ satisfies the following conditions:
(1) $\varphi \in C^1(\overline{\Omega}) \cap C^2(\Omega)$; (2) $\varphi(x) \geq 0$ $(x \in \Omega)$, $\varphi(x) = 0$ $(x \in \partial\Omega)$;
(3) $\Delta\varphi + \lambda e^{\varphi} \geq 0$ $(x \in \Omega)$ (resp. $\Delta\varphi + \varphi^p \geq 0$ $(x \in \Omega)$).
Then, by the maximum principle, it follows that $u(x,t) > 0$, $u_t(x,t) > 0$ for $(x,t) \in \Omega \times (0,T)$.

Suppose that the solution $u(x,t)$ blows up as $t \to T$ $(< \infty)$, i.e., $\lim_{t \to T} \max_{x \in \Omega} u(x,t) = \infty$. We are interested in the behabior of the limit function $\lim_{t\to T} u(x,t)$ near a blow-up point. (A point $x\in\overline{\Omega}$ is ,by definition, a blow-up point if there exists a sequence $(x_n, t_n)\in \Omega\times(0,T)$ such that $x_n\to x$, $t_n\to T$, and $u(x_n, t_n) \to\infty$ as $n \to\infty$.)

If Ω is a ball with center 0 and if φ satisfies the conditions (1), (2), (3) and if

(4) φ is radially symmetric with $\varphi_r \le 0$,

then a blow-up point is a single point $x = 0$ ([5]) and the following estimates are known. Upper estimates for (IBVP1) ([4]): There exists a constant C such that

(5) $u(x,t) \le -2 \ln |x| + \ln |\ln |x|| + C$ $(|x| > 0)$.

The inequality (5) cannot be improved as follows :

$$u(x,t) \le -2 \ln |x| + C.$$

In other words there does not exist such a constant C (see [3]).

Upper estimates for (IBVP2) ([5]): For any $0 < q < p$, there exists a constant C_q such that

$$u(x,t) \le C_q(1/|x|)^{2/(q-1)} \qquad (|x| > 0).$$

Lower estimates for (IBVP1) ([3]): Let $n \ge 3$. Then there exists $r_1 > 0$ such that

$$\lim_{t\to T} u(x,t) > \ln (2(n-2)/(\lambda|x|^2)) \qquad (0 < |x| < r_1).$$

Lower stimates for (IBVP2) ([3]): Let $n \ge 3$ and let $p>n/(n-2)$. Then there exists $r_1 > 0$ such that ($\beta = 1/(p-1)$)

$$\lim_{t\to T} u(x,t) > (-4\beta(\beta+(2-n)/2)/|x|^2)^{\beta} \qquad (0 < |x| <r_1).$$

We shall investigate lower estimates for solutions to (IBVP1,2) in the case that a domain in $\mathbb{R}^n$ ($n \geq 2$) is not necessary a ball. Our results on lower estimates are as follows.

Theorem 1. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that φ satisfies the conditions (1), (2), (3). Suppose that the solution $u(x,t)$ to (**IBVP1**) blows up at a point $x_0 \in \Omega$ as $t \to T$. Then there exists $r_1 > 0$ and the following inequality holds. For any r with $0 < r \leq r_1$

$$(6) \qquad \lim_{t\to T} \{1/(2\pi)\int_0^{2\pi} e^{u(x_0+(r\cos\theta,\, r\sin\theta),\, t)/2} d\theta\}^2 \quad > \quad 2\lambda^{-1}\, r^{-2} .$$

As a direct consequence of Theorem 1 we have

Corollary 2. Let Ω be a ball with center 0 in $\mathbb{R}^2$. Assume that $\varphi(x)$ is radially symmetric and satisfies the conditions (1), (2), (3). Suppose that the solution $u(x,t)$ to (**IBVP1**) blows up at $x = 0$ as $t \to T$. Then there exists $r_1 > 0$ and the following inequality holds. For any x with $0 < |x| < r_1$

$$(7) \qquad \lim_{t\to T} u(x,t) \quad > \quad -2\ \ln |x| \quad + \quad \ln(\, 2\lambda^{-1}).$$

Next we state the results on (**IBVP2**).

Theorem 3. Let Ω be a bounded convex domain in $\mathbb{R}^n$ $(n\geq2)$ with smooth boundary $\partial\Omega$. If x_0 is a blow-up point with a blow-up time T, then for any $k > 0$, there exists $r_k > 0$ such that

$$\lim_{t\to T} u(x,t) \geq k\left(1/|x - x_0| \right)^{2/(p-1)}$$

for any x with $0 < | x - x_0 | < r_k$.

Lastly we state results on the Hausdorff dimension of the set of blow-up points of the solution to (IBVP2). Let

$$\Gamma \equiv \{ \gamma : \text{the solution } u(x,t) \text{ blows up in } L^\gamma \text{ as } t \to T \}$$

and set $\gamma_0 \equiv \inf \Gamma$.

By the result of Giga and Kohn [9], it follows that $n(p-1)/2 \in \Gamma$ and so $\gamma_0 \leq n(p-1)/2$.

Theorem4. Let Ω be a bounded convex domain with smooth boundary $\partial\Omega$. Then the Hausdorff dimension D_H of the set of blow-up points for (IBVP2) satisfies the following inequality

$$D_H \leq -2\,\gamma_0/(p-1) + n.$$

For the proof of Theorem 1 we use Bandle's mean value theorem, which is proved by using Bol's isoperimetric inequality. In Section 2 we recall her result and prove Theorem 1. For the proof of Theorem 3, Giga and Kohn's results ([10]) play an essential role. As a consequence of Theorem 3 we obtain Theorem 4.

Section 2. Proof of Theorem 1.

For the proof we use Bandle's mean value theorem. Before proceeding to the proof we recall her result.

Let D be a domain in $\mathbb{R}^2$, and let v satisfies a differential inequality $\Delta v + \lambda e^v \geq 0$ in D. Let $B(r, x_0) \subset D$ be the disk of radius r with center $x_0 \in D$. We define a Riemann metric $d\sigma$ in D by $d\sigma^2 = e^{v(x)}(dx_1^2 + dx_2^2)$. Then the length of $\partial B(r, x_0)$ and the area of $B(r, x_0)$ are respectively

$$\ell(r) \equiv \int_0^{2\pi} e^{v(r,\theta)/2} r d\theta \qquad \text{and} \qquad a(r) \equiv \int_0^{2\pi} \int_0^r e^{v(r,\theta)} r dr d\theta$$

where we write $v(r,\theta)$ for $v(x)$ ($x = x_0 + (r\cos\theta, r\sin\theta)$).

Bol's isoperimetric inequality says that

$$\ell(r)^2 \geq 4\pi\, a(r) - (\lambda/2)\, a(r)^2. \tag{8}$$

By the Schwarz inequality we have $2\pi r\, a_r(r) \geq \ell^2$. Hence this together with (8) yields a differential inequality

$$2\pi r\, a_r(r) \geq 4\pi\, a(r) - (\lambda/2)\, a(r)^2 .$$

The equality holds if and only if

$$a(r) = a_0(r) \equiv \pi b r^2/(1+\lambda b r^2/8) \qquad (b : \text{a positive parameter}).$$

Note that $a_0(r)$ is the area of $B(r, x_0)$ with respect to a Riemann metric $d\sigma_0^2 = e^{v_0(x)}(dx_1^2 + dx_2^2)$, where

$$v_0(x) = \ln (b/(1 + \lambda b r^2/8)^2) \tag{9}$$

and it satisfies $\Delta v_0 + \lambda e^{v_0} = 0.$

We denote the length of $\partial B(r, x_0)$ with respect to $d\sigma_0$ by $\ell_0(r)$.

Now we state Bandle's mean value theorem ([1, 2]).

Lemma 1 (Bandle's mean value theorem).

Let v be as above. Let $B(r_1, x_0) \subset D$. Suppose that there exists $b > 0$ such that

(10) $\ell(r_1) = \ell_0(r_1)$.

Then if $\lambda\, a(r_1) \le 4\pi$, then

(i) $a(r) \le a_0(r)$ for $0 < r \le r_1$,

(ii) $v(0) \le \ln b$.

For the proof see [1].

Lemma 2. Let $r_1 > 0$ be fixed. Suppose that $\lambda\, a(r_1) \le 4\pi$. Then if

$$e^{\overline{v}(r_1)} \equiv ((1/2\pi)\int_0^{2\pi} e^{v(r_1,\theta)/2} d\theta\,)^2 < 2/(\lambda r_1^2\,),$$

then there exists $b > 0$ such that (10) holds and

(11) $\lambda\, a_0(r_1) < 4\pi$.

Proposition. Let $u(x, t)$ be a continuous function in $\Omega\times(0, T_0)$. Assume that for each $0 \le t < T_0$, $u(x, t)$ satisfies a differential inequality $\Delta u + \lambda e^u \ge 0$ in Ω. Suppose that $\lambda\, a(r_1, 0) < 4\pi$ and that for each $0 \le t < T_0$ $e^{\overline{u}(r_1,t)} < 2/(\lambda r_1^2)$. Then the followings hold for each $0 \le t < T_0$

(i) $\lambda\, a(r_1, t) < 4\pi$, and

(ii) $e^{u(x_0, t)} \le 4\, e^{\overline{u}(r_1, t)}$,

where $a(r_1, t) = \int_{B(r_1, x_0)} e^{u(x, t)}\, dx$,

(12) $e^{\overline{u}(r_1, t)} \equiv ((1/2\pi)\int_0^{2\pi} e^{u(r_1, \theta, t)/2}\, d\theta\,)^2$,

$$u(r_1, t, \theta) = u(x, t) \quad (x = x_0 + (r\cos\theta, r\sin\theta)).$$

Proof of Theorem 1. Since $u_t > 0$, we have differential inequalities $\Delta u + \lambda e^u > 0$. Let $r_1 > 0$ be so small that

(i) $\lambda a(r_1, 0) < 4\pi$; (ii) $e^{\overline{u}(r_1, 0)} < 2/(\lambda r_1^2)$.

We prove that there exists $T_1 < T$ such that $e^{\overline{u}(r_1, T_1)} = 2/(\lambda r_1^2)$. Suppose that there does not exist such a T_1. Then we have for $0 \le t < T$, $e^{\overline{u}(r_1, t)} < 2/(\lambda r_1^2)$. Hence by (ii) of Proposition we have

$$u(x_0, t) \le \ln \overline{u}(r_1, t) + 2 \ln 2 \qquad (0 \le t < T),$$

and so we have

$$u(x_0, t) \le -2 \ln r_1 + \ln(8/\lambda) \qquad (0 \le t < T),$$

contradicting the fact that x_0 is a blow-up point and that T is a blow-up time.

Proof of Proposition. First we prove that (i) holds for $0 \le t < T_0$. Suppose that it does not hold, then there exists $0 < T_1 < T_0$ such that $\lambda a(r_1, t) < 4\pi$ $(0 \le t < T_1)$, and

(13) $\qquad \lambda a(r_1, T_1) = 4\pi$.

Since $e^{\overline{u}(r_1, T_1)} < 2/(\lambda r_1^2)$, we get by Lemmas 1,2 that $\lambda a(r_1, T_1) < 4\pi$. This contradicts (13).

Next we prove that (ii) holds for $0 \le t < T_0$. Since

(14) $\qquad (\lambda r_1^2/2) e^{\overline{u}(r_1, t)} < 1,$

there exists $b = b(r_1, t)$ such that $\ell(r_1, t) = \ell_0(r_1, t)$, where $\ell(r_1, t)$ is the length of $\partial B(x_0, r_1)$ with respect to a Riemann metric $d\sigma^2 = e^{u(x, t)}(dx_1^2 + dx_2^2)$ (it equals to $2\pi r_1 e^{\overline{u}(r_1, t)/2}$) and $\ell_0(r_1, t) = 2\pi r_1 b^{1/2} / (1 + \lambda b r_1^2/8)$.

Indeed b is a solution of a quadratic equation

$$A^2Cb^2 + (2AC - 1)b + C = 0,$$

where $A = \lambda r_1^2/8$; $C = e^{\overline{u}(r_1, t)}$.

We choose $b = (1 - 2AC - \sqrt{1 - 4AC})/(2A^2C)$,

Hence by (i) and by Lemma 1 we get

$$(15) \qquad u(x_0, t) \leq \ln b(r_1, t).$$

On the other hand, we have by (14), $b(r_1, t) < 4C$. Hence by (15) we obtain

$$u(x_0, t) < 2 \ln 2 + \overline{u}(r_1, t).$$

Thus we have proved Proposition.

Proof of Lemma 2. Set $A = \lambda r_1^2/8$ and $C = e^{\overline{u}(r_1, t)}$. Since $4AC < 1$, there exists b such that $\ell(r_1) = \ell_0(r_1)$. We choose $b = (1 - 2AC - \sqrt{1 - 4AC})/ (2A^2C)$. Then an inequality

$$\lambda a_0(r_1) \quad (= \pi\lambda b r_1^2/(1+\lambda b r_1^2/8)) < 4\pi$$

holds if and only if $Ab < 1$. On the oher hand this inequality holds if and only if $(1 - X -\sqrt{1 - 2X})/ X < 1$ $(X = 2AC)$ i.e., $4AC < 1$. Thus we have proved Lemma 2.

Section 3. Proofs of Theorems 3,4.

Proof of Theorem 3. We prove that for any $k > 0$ there exists $r_k > 0$ such that for $0 < |x| < r_k$ ($x_0 = 0$)

$$\lim_{t \to T} u^{p-1}(x, t) > k/|x|^2 .$$

Let $k>0$ be fixed. Suppose that there does not exist such r_k. Then there exists a sequence $\{ x_\ell \}$ with $x_\ell \to 0$ such that

$$\lim_{t \to T} u^{p-1}(x_\ell, t) \leq k/|x_\ell|^2 .$$

Since $u_t > 0$, it follows that

(16) $\qquad |x_\ell|^2 u^{p-1}(x_\ell, t) \leq k \qquad$ for $t > 0$.

Choose $\varepsilon > 0$ with

(17) $\qquad \varepsilon < \beta/k \qquad (\beta=1/(p-1))$ and set $t_\ell \equiv T - \varepsilon|x_\ell|^2$. Then by (16) we get

(18) $\qquad \varepsilon^{-1}(T-t_\ell)u^{p-1}(x_\ell, t_\ell) \leq k.$

On the other hand by the result of Giga and Kohn it follows that

$$(T-t_\ell)u^{p-1}(x_\ell, t_\ell) \to \beta.$$

Hence by (18) we get $\beta \leq \varepsilon k$. This contradicts (17).

Proof of Theorem 4. Before proceeding to the proof we recall the definition of the Hausdorff dimension. Let X be a metric space and let Y be a subset of X. For $\varepsilon > 0$ and $D > 0$ we set

$$\mu_{D,\varepsilon}(Y) \equiv \inf_{\text{diam } B_j < \varepsilon} \sum_{Y \subset \cup B_j} (\text{diam } B_j)^D ,$$

$$\mu_D(Y) \equiv \lim_{\varepsilon \downarrow 0} \mu_{D,\varepsilon}(Y) = \sup_{\varepsilon > 0} \mu_{D,\varepsilon}(Y).$$

Then $\mu_D(Y)$ is called the D-dimensional Hausdorff measure of Y. The Hausdorff dimension of Y is defined as follows:

$\inf \{ D : \mu_D(Y) < \infty \} \quad (= \inf \{ D : \mu_D(Y) = 0 \}).$

We prove the following proposition. Theorem 4 is an immediate consequence of it.

Proposition. Suppose that u does not blow-up in L^γ as $t \to T$, i.e., $\lim_{t\to T} \| u \|_{L^\gamma} < \infty$. Let $D \equiv -2\gamma/(p-1) + n$. Then the D-dimensional Hausdorff measure μ_D of the blow-up points satisfies

(19) $$\mu_D \leq c \lim_{t\to T} \| u \|_{L^\gamma}^\gamma \qquad (\text{ c: some constant}).$$

Proof of Proposition. Let $\varepsilon > 0$ be fixed. Let x_0 be a blow-up point. Since Ω is convex, it follows that $x_0 \in \Omega$ ([5]). Hence by Theorem 3, there exists $0 < r_{x_0} \leq 2/3\ \varepsilon$ such that

$$B(x_0, r_{x_0}) \equiv \{ x : | x - x_0 | < r_{x_0} \} \subset \Omega ,$$

$$\lim_{t\to T} u(x,t) \geq | x - x_0 |^{-2/(p-1)}, \qquad x \in B(x_0, r_{x_0}).$$

Since the set of blow-up points is compact ([5]), there exists a finite subset $\{ x_j \}$ of the blow-up points and r_j with $0 < r_j \leq 2/3\ \varepsilon$ such that

$$B(x_i, 3/2\ r_i) \subset \Omega ;$$

(20) $$B(x_i, r_i) \cap B(x_j, r_j) = \Phi \quad (i \neq j);$$

$$\{ \text{blow-up points} \} \subset \bigcup_i B(x_i, 3/2\ r_i) ;$$

(21) $$\lim_{t\to T} u^\gamma(x,t)dx \geq | x - x_0 |^{-2\gamma/(p-1)}, \quad x \in B(x_i, r_i).$$

By (21) we get

(22) $$\int_{|x-x_i|\leq r_i} \lim_{t\to T} u^\gamma(x,t)dx \geq \iint_{0\leq r\leq r_i} r^{-2\gamma/(p-1)} r^{n-1} dr d\omega \equiv C r_i^D .$$

Hence by (20), (22) we get

$$\sum_i (3/2\ r_i)^D \le (3/2)^D C^{-1} \sum_i \int_{|x-x_i|\le r_i} \lim_{t\to T} u^\gamma(x,t)\,dx$$
$$\le (3/2)^D C^{-1} \int_\Omega \lim_{t\to T} u^\gamma(x,t)\,dx$$
$$= (3/2)^D C^{-1} \lim_{t\to T} \int_\Omega u^\gamma(x,t)\,dx < \infty .$$

Since C is independent of ε, we obtain (19).

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